Theoretical aspects of high energy elastic nucleon scattering

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Abstract

The eikonal model must be denoted as strongly preferable for the analysis of elastic high-energy hadron collisions. The given approach allows to derive corresponding impact parameter profiles that characterize important physical features of nucleon collisions, e.g., the range of different forces. The contemporary phenomenological analysis of experimental data is, however, not able to determine these profiles unambiguously, i.e., it cannot give the answer whether the elastic hadron collisions are more central or more peripheral than the inelastic ones. However, in the collisions of mass objects (like protons) the peripheral behavior of elastic collisions should be preferred.

1 Coulomb-hadronic interference

The first attempt to determine the complete elastic scattering amplitude $F^{C+N}(s,t)$ for high energy collisions of charged nucleons has been made by Bethe [1]:

$$F^{C+N}(s,t) = F^C(s,t)e^{i\alpha\Psi(s,t)} + F^N(s,t) \tag{1}$$

where $F^C(s,t)$ is Coulomb amplitude (known from QED), $F^N(s,t)$ - elastic hadronic amplitude, $\alpha\Psi(s,t)$ - real relative phase between Coulomb and hadronic scattering and $\alpha = 1/137.036$ is the fine structure constant. This relative phase has been specified by West and Yennie [2] using the Feynman diagram technique (one photon exchange) as

$$\Psi_{WY}(s,t) = -\ln\frac{-s}{t} - \int_{-4p^2}^{0}\frac{dt'}{|t'-t|}\left[1 - \frac{F^N(s,t')}{F^N(s,t)}\right]; \tag{2}$$

p representing the value of the incident momentum in the centre-of-momentum system. By their construction (see Eq. (2), the phase $\Psi_{WY}(s,t)$ is to be real.

However, it has been shown in Ref. [3] that this requirement can be fulfilled only provided the phase of the elastic hadronic amplitude $\zeta^N(s,t)$ defined in our case as

$$F^N(s,t) = i|F^N(s,t)|e^{-i\zeta^N(s,t)}, \tag{3}$$

is t independent at all kinematically allowed values of t. Rigorous proof has been given for $|\zeta^N(s,t)| < \pi$, which is fulfilled practically in all standard phenomenological models leading to the central distribution of elastic processes in impact parameter space. It is not yet clear if it holds also for $|\zeta^N(s,t)| < 2\pi$ corresponding to peripheral behavior; see Fig. 1 where both the types of phase t dependences are represented. The corresponding t dependences of imaginary parts of complex relative phases $\alpha\Psi_{WY}(s,t)$ are shown in Fig. 2.

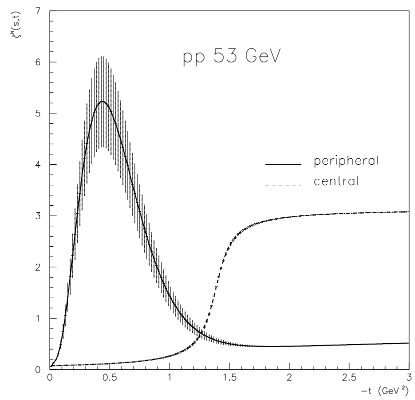

Figure 1: t dependences of the hadronic phases $\zeta^N(s,t)$ leading to the central and peripheral behaviors of elastic pp scattering.

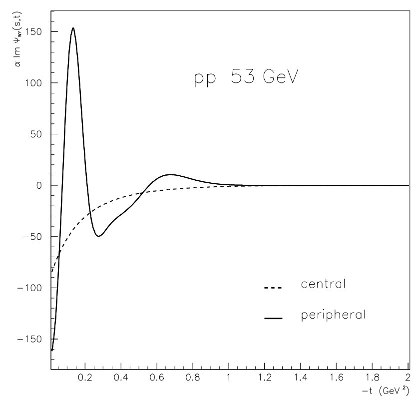

Figure 2: t dependences of imaginary parts of the complex WY phases $\alpha\Psi_{WY}(s,t)$ corresponding to hadronic phases from Fig. 1.

There is, however, no reason why the elastic hadronic phase should be independent of t variable at all measured values of t. Thus the other approach different from the West and Yennie formalism and based on the eikonal approach should be preferred - see, e.g., Ref. [4]. In such a case the complete elastic scattering amplitude $F^{C+N}(s,t)$ is related to the complete eikonal $\delta^{C+N}(s,b)$ with the help of Fourier-Bessel (FB) transformation

$$F^{C+N}(s,q^2=-t) = \frac{s}{4\pi i}\int\limits_{\Omega_b} d^2b e^{i\vec{q}.\vec{b}}\left[e^{2i\delta^{C+N}(s,b)}-1\right], \tag{4}$$

where Ω_b is the two-dimensional Euclidean space of the impact parameter $\vec{b}$. Mathematically consistent use of FB transformation requires, of course, the existence of reverse transformation. However, at finite energies the amplitude $F^{C+N}(s,t)$ is defined in a finite region of t only. Thus the consistent application of formula (4) requires to take into account also the values of elastic amplitude

from unphysical region where the elastic hadronic amplitude is not defined; for details see Refs. [5]. This issue has been discussed by Islam [6] who has shown that the problem may be solved in a unique way by continuing analytically the elastic hadronic amplitude $F^N(s,t)$ from physical to unphysical region of t ; see also Ref. [8].

The common influence of both the Coulomb and elastic hadronic scattering then can be described by complete eikonal which is formed by the sum of both the Coulomb $\delta^C(s,b)$ and hadronic $\delta^N(s,b)$ eikonals. Then the complete amplitude (valid at any s and t) can be finally written [4] as

$$F^{C+N}(s,t) = \pm\frac{\alpha s}{t} f_1(t) f_2(t) + F^N(s,t)\left[1 \mp i\alpha G(s,t)\right], \tag{5}$$

where

$$G(s,t) = \int\limits_{-4p^2}^{0} dt' \left\{ \ln\left(\frac{t'}{t}\right) \frac{d}{dt'}\left[f_1(t')f_2(t')\right] + \frac{1}{2\pi}\left[\frac{F^N(s,t')}{F^N(s,t)} - 1\right] I(t,t') \right\}, \tag{6}$$

and

$$I(t,t') = \int\limits_{0}^{2\pi} d\Phi'' \frac{f_1(t'')f_2(t'')}{t''}, \quad t'' = t + t' + 2\sqrt{tt'}\cos\Phi''. \tag{7}$$

Here the two form factors $f_1(t)$ and $f_2(t)$ reflect the spatial structure of colliding nucleons and should describe it in a sufficiently broad interval of t. As the Coulomb amplitude $F^C(s,t)$ is known from QED the complete amplitude is determined practically by the t dependence of the hadronic amplitude $F^N(s,t)$.

2 Impact parameter picture of elastic nucleon scattering

As it has been already mentioned the mathematically consistent use of FB transformation introducing the impact parameter representation $h_{el}(s,b)$ of elastic scattering amplitude $F^N(s,t)$ requires its definition also in the unphysical region of t, i.e., for $t < t_{min} = -s + 4m^2$ (m being the nucleon mass). The function $h_{el}(s,b)$ must be, therefore, subdivided into two parts [5, 6]

$$\begin{aligned} h_{el}(s,b) &= h_1(s,b) + h_2(s,b) = \\ &= \frac{1}{4p\sqrt{s}} \int\limits_{t_{min}}^{0} dt\, F^N(s,t)\, J_0(b\sqrt{-t}) + \frac{1}{4p\sqrt{s}} \int\limits_{-\infty}^{t_{min}} dt\, F^N(s,t)\, J_0(b\sqrt{-t}). \end{aligned} \tag{8}$$

Similar expressions can be obtained also for the impact parameter representation $g_{inel}(s,b)$ of the inelastic overlap function $G_{inel}(s,t)$ introduced in Ref. [7].

The unitarity equation in the impact parameter space can be then written as [5, 6]

$$\Im h_1(s,b) = |h_1(s,b)|^2 + g_1(s,b) + K(s,b) \tag{9}$$

where the correlation function $K(s,b)$ is very small compared to the other functions appearing in Eq. (9) [8].

The total cross section and integrated elastic and inelastic cross sections then equal to

$$\sigma_{tot}(s) = 8\pi\int\limits_0^\infty bdb\, \Im h_1(s,b);\ \sigma_{el}(s) = 8\pi\int\limits_0^\infty bdb\, |h_1(s,b)|^2;\ \sigma_{inel}(s) = 8\pi\int\limits_0^\infty bdb\, g_1(s,b). \tag{10}$$

Eqs. (10) are valid provided

$$\int\limits_0^\infty bdb\, \Im h_2(s,b) \ = \ 0, \qquad \int\limits_0^\infty bdb\, g_2(s,b) \ = \ 0. \tag{11}$$

The functions $\Im h_1(s,b)$, $g_1(s,b)$ and $|h_1(s,b)|^2$ represent then the impact parameter profiles; they describe the density of interactions between two colliding nucleons in dependence on their mutual distance. The first two oscillate at grater values of b, but their mean squared values characterizing the mean ranges of corresponding forces acting between the colliding particles can be established directly from the elastic hadronic amplitude $F^N(s,t)$ [9]. For the mean squared value of elastic impact parameter it has been derived

$$\langle b^2(s)\rangle_{el} \ = \ 4\,\frac{\int\limits_{t_{min}}^0 dt\, |t| \left(\frac{d}{dt}|F^N(s,t)|\right)^2}{\int\limits_{t_{min}}^0 dt\, |F^N(s,t)|^2} + 4\,\frac{\int\limits_{t_{min}}^0 dt\, |t|\, |F^N(s,t)|^2 \left(\frac{d}{dt}\zeta^N(s,t)\right)^2}{\int\limits_{t_{min}}^0 dt\, |F^N(s,t)|^2}. \tag{12}$$

Similarly the total and inelastic mean squared values equal to

$$\langle b^2(s)\rangle_{tot} \ = 2B(s,0); \quad \langle b^2(s)\rangle_{inel} \ = \ \frac{\sigma_{tot}(s)}{\sigma_{inel}(s)}\langle b^2(s)\rangle_{tot} \ - \ \frac{\sigma_{el}(s)}{\sigma_{inel}(s)}\langle b^2(s)\rangle_{el}. \tag{13}$$

Here the diffractive slope is defined as

$$B(s,t) = \frac{d}{dt}\left[\ln\frac{d\sigma^N}{dt}\right] = \frac{2}{|F^N(s,t)|}\frac{d}{dt}|F^N(s,t)|. \tag{14}$$

3 Impact parameter profiles for *pp* scattering at 53 GeV

Basic results concerning the analysis of *pp* elastic scattering data at energy of 53 GeV at the ISR [10] based on the eikonal approach have been published in Ref. [4]. Here we will mention only the results related to the impact parameter profiles.

In the quoted paper we have used the formulas (5)-(7) for the complete elastic scattering amplitude $F^{C+N}(s,t)$ generating the differential cross section

$$\frac{d\sigma_{el}}{dt} = \frac{\pi}{sp^2}|F^{C+N}(s,t)|^2. \tag{15}$$

The elastic hadronic amplitude, i.e., its modulus and phase defined in Eq. (3), has been conveniently parameterized in order to describe the pp elastic scattering as central as well as peripheral process. While the t dependence of the modulus can be almost unambiguously determined from the data the phase can be only partially constrained. Both the possible alternatives (central and peripheral) have been presented in Ref. [4]. The t dependences of the corresponding shapes of the hadronic phase $\zeta^N(s,t)$ are shown in Fig. 1.

Once the elastic hadronic amplitude $F^N(s,t)$ has been specified it has been possible to determine corresponding impact parameter profiles together with their statistical errors with the help of FB transformation. And it has been also possible to determine the RMS values of the total, elastic and inelastic profiles with the help of Eqs. (12) and (13) for the central as well as the peripheral pictures of elastic pp scattering. Their shapes corresponding to peripheral behavior are shown in Fig. 3 (for the central case see Ref. [3]); all RMS values are included in Table 1. In the central picture the elastic RMS is much lower than the inelastic one. This result agrees with the result of Miettinen [11]. It means that the protons in 'head-on' collisions should be rather transparent. In

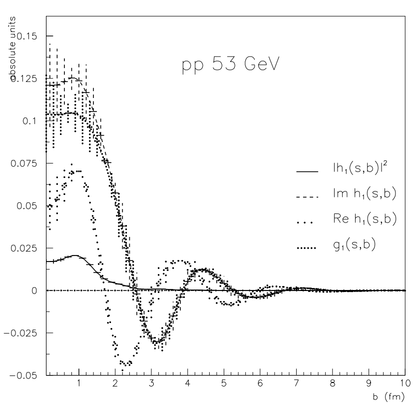

Figure 3: Oscillating peripheral profiles of elastic pp scattering at 53 GeV.

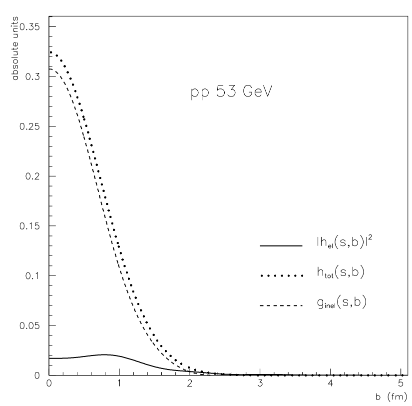

Figure 4: Positive peripheral profiles of elastic pp scattering at 53 GeV.

the peripheral case it is the second term in Eq. (12) that gives the significant contribution to the elastic RMS value. Some profiles corresponding to the peripheral case exhibit greater oscillations at higher impact parameter values b. However, the oscillations can be removed as it will be shown in Ref. [8]. It will be briefly described in the following.

It is necessary to construct actual (non-negative) pp profiles which correspond to the values of RMS derived with the help of Eqs. (12) and (13). The

Table 1: Root-mean-squares of impact parameters for pp collisions at 53 GeV.

elastic profile	$\sqrt{<b^2>_{tot}}$		$\sqrt{<b^2>_{el}}$		$\sqrt{<b^2>_{inel}}$
		modulus	phase	sum	
	[fm]	[fm]	[fm]	[fm]	[fm]
peripheral	1.033	0.676	1.671	1.803	0.772
central	1.028	0.679	$\sim 0.$	0.679	1.087

way how to do it is described in Refs. [8, 9]. It consists in adding an real function $c(s,b)$ to the both sides of the unitarity equation (9). All dynamical characteristics corresponding to elastic hadron scattering will be preserved if the function $c(s,b)$ appearing in Eqs. (17) fulfills some additional conditions.

Unitarity condition (9) may be then expressed as

$$h_{tot}(s,b) = |h_1(s,b)|^2 + g_{inel}(s,b) \tag{16}$$

where

$$h_{tot}(s,b) = \Im h_1(s,b) + c(s,b), \quad g_{inel}(s,b) = g_1(s,b) + K(s,b) + c(s,b) \tag{17}$$

are non-negative functions for all b. The function $h_{tot}(s,b)$ must be positively semidefinite (and monotony decreasing function) at all values of b. The oscillating function $c(s,b)$ is required to cancel the oscillations from the total and inelastic profiles. The elastic peripheral profile $|h_1(s,b)|^2$ will remain unchanged. And integrated inelastic cross section is preserved if the function $c(s,b)$ fulfills the conditions

$$\int\limits_0^\infty b\, db\, c(s,b) = 0; \qquad \int\limits_0^\infty b^3\, db\, c(s,b) \; = \; 0. \tag{18}$$

These two conditions represent the integral conditions limiting the shape of the function $c(s,b)$. According to Islam's approach [6] this function may be identified with the $\Im h_2(s,b)$ for which the conditions (18) are to be required. However, in the standard approach the function $c(s,b)$ can be hardly determined analytically. The best way at the present seems to specify it in a numerical way as it will be illustrated in the following. It may be expected that the total profile entering into modified unitarity condition (16) should be approximately of Gaussian type with the values that may be characterized by integral cross

Table 2: The values of integrated cross sections and of the total, elastic and inelastic RMS.

Quantity		Original values	New values
σ_{tot}	[mb]	42.864	42.872
σ_{el}	[mb]	7.479	7.479
$\sqrt{< b^2 >_{tot}}$	[fm]	1.0	1.028
$\sqrt{< b^2 >_{el}}$	[fm]	1.803	1.803
$\sqrt{< b^2 >_{inel}}$	[fm]	0.772	0.772
$\int_0^\infty bdb\ c(s,b)$	[fm^2]	-	0.029
$\int_0^\infty b^3 db\ c(s,b)$	[fm^4]	-	0.097

sections and by RMS values shown in Table 1. The elastic profile will remain unchanged. Under these assumptions the total profile shape can be defined (the s dependence being dropped) as $h_{tot}(b) \ = \ ae^{-\beta b^2}$. Using formulas 3.461 from Ref. [13] the corresponding integrals needed for calculation of the total cross section and total mean squared value can be analytically determined as

$$\int\limits_0^\infty b\ db\ a\ e^{-\beta b^2} \ = \ \frac{a}{2\beta}, \qquad \int\limits_0^\infty b^3\ db\ a\ e^{-\beta b^2} \ = \ \frac{a}{2\beta^2} \tag{19}$$

and the values of the constants a and β can be determined from experimentally established values. For the peripheral case of elastic pp scattering at energy of 53 GeV their values are: $a = 0.324$ and $\beta = 0.946$. The b dependence of the auxiliary function $c(s,b)$ is then determined with the help of the first equation from (17) where $\Im h_1(s,b)$ is taken from experimental analysis. And the second equation determines then the shape of the inelastic profile.

Knowing the shapes of the total and inelastic profiles together with the b dependence of the auxiliary function $c(s,b)$ the values of all the integrated cross sections and of all the mean squares can be verified numerically as can be seen from the Table 2. The new values are practically quite comparable with the original ones. Also the values of the integrals of function $c(s,b)$ (see Eqs. (18)) only slightly different from zero are shown in Table 2. The modified profiles are exhibited in Fig. 4. The new total and inelastic profiles are central while the elastic profile remains unchanged and is peripheral.

4 Conclusion

Some results concerning elastic nucleon collisions at high energies and based on the validity of optical theorem have been summarized; the eikonal model has been applied to. The approach suitable for the case of finite energies has been presented. It has been shown that elastic processes may be interpreted as peripheral.

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